Free-Space Bounded Plane-Wave Solutions to the Maxwell-Lorentz Equations

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(Received 23 September 1963; revised manuscript received 27 February 1964)

A relativistic fluid concept of electric charge is introduced, from which free-space laterally-bounded plane-wave solutions to the inhomogeneous Maxwell-Lorentz equations can be obtained in harmony with the particle-like nature of the radiation field.

INTRODUCTION

EXPERIMENTAL evidence supports the view that electric charge always occurs in nature in multiples of an indivisible unit of approximately 1.6×10^{-19} C. There is, on the other hand, no experimental evidence to contradict a view that this unit of charge is a net charge equal to the difference between positive and negative components. If this view is reservedly adopted, then neutral particles can be viewed as having equal positive and negative components. Such a view is not too difficult to adopt for particles having nonzero rest mass. It becomes somewhat difficult to adopt for photons, however, inasmuch as this would require the charge components to propagate with the speed of light, which would appear to violate special relativity. This apparent violation results, however, from the usual view that charge can be associated only with particles having nonzero rest mass. If we tentatively adopt the view that charge *can* be associated with zero-rest-mass particles, there is no violation of special relativity inasmuch as charge, unlike mass, is invariant with speed.

Let us investigate the consequences of assuming that charge can act like a distributed fluid propagating with the speed of light.

RELATIVISTIC CHARGE

From classical relativistic electrodynamics it is known that as a charged particle (of nonzero rest mass) approaches the speed of light, the electromagnetic fields of the particle, as viewed in the laboratory frame, approach those of an electromagnetic wave. If we let a differential test charge translate with exactly the speed of light, the charge sets up purely transverse electric and magnetic fields that are mutually perpendicular and related in magnitude by the impedance of free space, $(\mu/\epsilon)^{1/2}$, where μ and ϵ are the permeability and permittivity of free space, respectively. In this limiting case the electromagnetic fields of the charge are exactly those of a plane electromagnetic wave.

With the exception of physically unrealistic infinite plane waves, we are accustomed to visualizing electromagnetic waves in terms of electric field lines of force that always form closed loops. Wave solutions of this type are the only physically realistic ones possible from the homogeneous Maxwell's equations, obtained by setting the charge density $\rho=0$. We shall show that physically realistic waves containing charge, in which the electric field lines of force do *not* form closed loops, result naturally from considering solutions to the complete inhomogeneous Maxwell's equations, in which the charge density ρ is not set equal to zero.

Consider two differential elements of charge of opposite sign, $+dq$ and $-dq$, separated by a certain distance and translating parallel to each other in a direction perpendicular to the line joining the elements of charge. The electric and magnetic fields of the two charges will all lie in a plane passing through the charges, and perpendicular to the direction of propagation. Such a translating dipole constitutes a differential plane electromagnetic wave having zero net charge. It can readily be shown that the Lorentz force on each element of charge due to the fields of the other is exactly zero. The same is true for two differential elements of charge of the same sign. It follows that for *any* arbitrary continuous distribution of charge, all elements of which lie in a plane and propagate with the speed of light in a direction normal to the plane, the Lorentz force on every differential element of charge due to the fields of all the other charge elements is automatically zero.

Consider two parallel planes, each of which contains an arbitrary charge distribution, all elements of which are propagating with the speed of light in a direction normal to the planes. The charge in one plane is not subjected to any Lorentz force due to the fields of the charges in the other plane because the fields of the charges in both planes are purely transverse and lie in their respective planes. It follows that for *any* arbitrary three-dimensional distribution of charge, all elements of which are translating in parallel directions with the speed of light, there is zero Lorentz force on every differential element of the charge due to the fields of all the other charge elements. This fact can be used to advantage in setting up wave equations that are only first order in time and distance, as opposed to the usual second-order wave equations obtained from the homogeneous Maxwell's equations. (See Appendix.)

A three-dimensional distribution of charge, all elements of which are translating in parallel directions with the speed of light, constitutes a three-dimensional plane electromagnetic wave. Whereas free-space plane waves obtained from the homogeneous Maxwell's equations are uniform in the lateral direction out to infinity, plane waves containing distributed charge are in general nonuniform laterally. Solutions to the inhomogeneous

Maxwell's equations are possible that represent plane-wave packets that are effectively bounded in three dimensions. (See Appendix.) The physical size of such packets of electromagnetic energy remains invariant with time and distance, in accord with the particle-like property of the radiation field.

A question arises, however, as to the effect of *external* electric and magnetic fields upon such waves containing distributed charge. If the bounded plane-wave packets are to have the physical significance of representing photons, external fields must have no effect on the packets. It is not sufficient merely to have zero net charge for the packet. Every differential element of the distributed charge must not be affected by external fields, for otherwise the packet would be disrupted in violation of experimental observation.

Intuitively, it may seem as though external fields would indeed disrupt a packet by affecting positive and negative charge components oppositely. This intuitive answer arises from long experience with charged particles having nonzero rest mass however. We are dealing with an entirely different situation with zero rest-mass particles containing charge. For example, the electrodynamics of nonzero rest-mass particles is well understood for all particle velocities less than the speed of light. But the equations of motion of charge translating with the speed of light are completely unknown. It is a trivial matter to compute the Lorentz forces on a differential element of charge translating with the speed of light through a region containing external electromagnetic fields. But without knowledge of the equations of motion, a knowledge of the forces is, in itself, useless.

The presence of forces does not necessarily imply acceleration, even when dealing with nonzero rest-mass particles. Only unconstrained particles can be accelerated by forces. If it is simply postulated that charge translating with the speed of light is *constrained* to move in a straight line at the speed of light, then the equations of motion of the charge are, under all circumstances, simply

$$\dot{z}=c, \tag{1}$$

$$\dot{x}=0, \tag{2}$$

and

$$\dot{y}=0, \tag{3}$$

where the z axis is chosen in the direction of propagation of the charge. If photons are indeed bounded plane electromagnetic wave packets, as is being suggested, then Eqs. (1) through (3) would appear to be the required equations of motion consistent with experimental observation.

If charge translating with the speed of light is constrained to move in a straight line at the speed of light under all circumstances, as postulated, there appears to be no way to detect the charge experimentally. Thus, either charge cannot translate with the speed of light, or it can but only under conditions such that it cannot be detected. The usefulness in assuming that it can is that solutions to the Maxwell-Lorentz equations can be obtained that represent bounded plane-wave packets whose physical size is invariant with time elapsed or distance traversed, in harmony with the particle-like nature of the radiation field.

Lanczos[1] has recently shown that electromagnetic waves can propagate with low leakage in a space that is highly curved in a Riemannian sense in submicroscopic domains, and that such a space is almost Minkowskian in the macroscopic in harmony with the requirements of special relativity. Lanczos thus justifies Einstein's photon hypothesis on a classical field-theoretic basis. The present paper can be considered to be an alternative justification.

THE LARGE DENSITY LIMIT

If a photon is a bounded plane-wave packet containing distributed charge, it is evidently required that such a packet be the fundamental mode generated (and absorbed) by a single electron. We have, then, a picture of radiation wherein the number of wave packets generated equals the number of electrons having generated such packets. If a sufficiently large number of electrons in close proximity generate a large number of wave packets over a short time interval, such packets will overlap each other near the source. Under these circumstances we may inquire as to the effect such overlapping will have on the "apparent" charge density in the waves. It can be shown that as the density of overlapping packets is increased, the "apparent" charge density will decrease because the positive components of some packets will tend to cancel the negative components of other overlapping packets. In the limit, as the packet density approaches infinity, the "apparent" charge density at any point approaches zero.

Thus, it can be concluded that the wave solutions to the homogeneous Maxwell's equations, wherein the charge density is set equal to zero, is the large quantum number approximation. This is simply the correspondence principle applied to the radiation field.

CONCLUSIONS

If charge can act like a continuous fluid translating with the speed of light, such an arrangement constitutes a plane electromagnetic wave that can be effectively bounded in the lateral direction. If such relativistic charge is constrained to translate in uniform rectilinear motion at the speed of light, unaffected by external electromagnetic fields, bounded plane-wave packets containing such charge may constitute a classical description of photons.

APPENDIX

The Maxwell-Lorentz equations for ρ translating in the $+z$ direction with the speed of light under the

[1] C. Lanczos, J. Math. Phys. 4, 951 (1963).

conditions of zero Lorentz force on each differential element of ρ can be written (for $E_z=B_z=0$),

$$[(\nabla\times\mathbf{H})_z=\rho c],\qquad \frac{\partial(rB_\varphi)}{\partial r}-\frac{\partial B_r}{\partial\varphi}=\left(\frac{\mu}{\epsilon}\right)^{\frac{1}{2}}r\rho\,, \tag{4}$$

$$[\nabla\cdot\mathbf{B}=0],\qquad \frac{\partial(rB_r)}{\partial r}+\frac{\partial B_\varphi}{\partial\varphi}=0\,, \tag{5}$$

$$\left[(\nabla\times\mathbf{H})_\varphi=\frac{\partial D_\varphi}{\partial t}\right],\qquad \frac{\partial B_r}{\partial z}=-\frac{1}{c}\frac{\partial B_r}{\partial t}\,, \tag{6}$$

$$\left[(\nabla\times\mathbf{E})_\varphi=-\frac{\partial B_\varphi}{\partial t}\right],\qquad \frac{\partial B_\varphi}{\partial z}=-\frac{1}{c}\frac{\partial B_\varphi}{\partial t}\,, \tag{7}$$

where the relations

$$D_r=(\epsilon/\mu)^{1/2}B_\varphi$$

and

$$D_\varphi=-(\epsilon/\mu)^{1/2}B_r,$$

obtained from the Lorentz force equation, have been used. Equations (6) and (7) have general solutions of the form

$$B_r=A_1f_1(ct-z) \tag{8}$$

and

$$B_\varphi=A_2f_2(ct-z), \tag{9}$$

where A_1 and A_2 are either constants or are functions of r and/or φ, but not z or t, and f_1 and f_2 are arbitrary functions of $(ct-z)$.

Equation (5) guarantees that $f_1=f_2$. Therefore, we may write as general solutions

$$B_r=B_1R_1\Phi_1f(ct-z) \tag{10}$$

and

$$B_\varphi=B_2R_2\Phi_2f(ct-z), \tag{11}$$

where B_1 and B_2 are constants, R_1 and R_2 are arbitrary functions of r, and Φ_1 and Φ_2 are arbitrary functions of φ. Equation (5) now guarantees that $B_1=B_2$. Also, from Eq. (5),

$$\Phi_1=-d\Phi_2/d\varphi \tag{12}$$

and

$$R_2=d(rR_1)/dr. \tag{13}$$

Therefore,

$$B_r=-BR(d\Phi/d\varphi)f(ct-z) \tag{14}$$

and

$$B_\varphi=B[d(rR)/dr]\Phi f(ct-z), \tag{15}$$

where B is a constant $(=B_1=B_2)$, R is any arbitrary function of r, and Φ is any arbitrary function of φ.

From Eq. (4), the required ρ is found to be

$$\rho=\left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2}}\frac{1}{r}B\left\{(\Phi)\left(d\left[r\frac{d(rR)}{dr}\right]\Big/dr\right)+R\frac{d^2\Phi}{d\varphi^2}\right\}f(ct-z). \tag{16}$$

For example, by setting $\Phi=\cos\varphi$,

$$R=r^2/r_0^2\exp(-r^2/r_0^2),$$

and

$$f(ct-z)=\exp[-(ct-z)^2/z_0^2]\sin[2\pi/\lambda(ct-z)],$$

an arbitrary "bounded" wave packet is obtained that is free of singularities or discontinuities. Such a packet has zero net charge and finite energy.

ACKNOWLEDGMENTS

The author wishes to express his gratitude to Philip Rice, Marshall C. Pease, Casper W. Barnes, Jr., Marshall Sparks, Leo Young, Richard S. Muller, and Burnell V. Dore for many stimulating discussions concerning the ideas presented in this paper.